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The Jacobi–Stirling numbers

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ABSTRACT

The Jacobi–Stirling numbers were discovered as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression. Specifically, these numbers are the coefficients of integral composite powers of the Jacobi expression in Lagrangian symmetric form. Quite remarkably, they share many properties with the classical Stirling numbers of the second kind which are the coefficients of integral powers of the Laguerre differential expression. In this paper, we establish several properties of the Jacobi–Stirling numbers and its companions including combinatorial interpretations, thereby extending and supplementing known recent contributions to the literature.

1. Introduction

The Jacobi–Stirling numbers, defined for $n, j \in \mathbb{N} := \{1, 2, 3, \ldots\}$ by

$$\left\{ {n \atop j} \right\}_{\alpha,\beta} := \sum_{r=0}^{j} (-1)^{r+j} \frac{(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)[r(r+\alpha+\beta+1)]^n}{r!(j-r)!\Gamma(\alpha+\beta+j+r+2)}, \tag{1.1}$$

were discovered in 2007 [10] in the course of the left-definite operator-theoretic study of the classical second-order Jacobi differential expression

$$\ell_{\alpha,\beta}[y](x) = \lambda y(x) \quad \bigl(x \in (-1, 1)\bigr), \tag{1.2}$$

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where

$$\ell_{\alpha,\beta}[y](x) := \frac{1}{w_{\alpha,\beta}(x)}\big(\big(-(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\big)' + k(1-x)^{\alpha}(1+x)^{\beta}y(x)\big)$$
$$= -\big(1-x^2\big)y''(x) + \big(\alpha - \beta + (\alpha+\beta+2)x\big)y'(x) + ky(x) \quad \big(x \in (-1,1)\big). \tag{1.3}$$

Here, we assume that $\alpha, \beta > -1$, k is a fixed, nonnegative constant, and $w_{\alpha,\beta}(x)$ is the Jacobi weight function defined by

$$w_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta} \quad \big(x \in (-1,1)\big). \tag{1.4}$$

The Jacobi–Stirling numbers have many properties similar to those of the classical Stirling numbers of the second kind, and yet their origin in differential operators is somewhat unusual. Indeed, generalizations of the Stirling numbers are usually defined by triangular recurrence relations of the form

$$S(n,k) = S(n-1,k-1) + W(n,k)S(n-1,k),$$

where $W(n,k)$ is a weight function. Note that the Stirling numbers of the second kind have $W(n,k) = k$, while the Stirling numbers of the first kind have $W(n,k) = n-1$. Most generalizations take $W(n,k)$ to be a linear function of n and/or k, but in our situation $W(n,k)$ is a quadratic function (see Theorem 3.1 below). The literature on generalizations of Stirling numbers is too vast for us to succinctly summarize it here, but the interested reader may wish to begin with [12]. Two excellent sources for the Stirling numbers of the second kind, and their properties, are the recently updated handbook [1], edited by Olver, Lozier, Boisvert, and Clark, as well as the classic text of Comtet [6, Chapter V]. Furthermore, Comtet further generalized the classical Stirling numbers in [7].

This manuscript may be viewed as a continuation of the combinatorial results obtained in [2,3,8]; each of these papers deals exclusively with various properties of the Legendre–Stirling numbers $\{PS_n^{(j)}\}$, a special case of the Jacobi–Stirling numbers. Indeed, by definition, $PS_n^{(j)} = \left\{ {n \atop j} \right\}_{0,0}$. The Jacobi–Stirling numbers have generated a significant amount of interest from other researchers in combinatorics. In this respect, we note that the Legendre–Stirling numbers appear in some recent work [5] related to the Boolean number of a Ferrers graph; these authors also show that there is an interesting connection between Legendre–Stirling numbers and the Genocchi numbers of the second kind. Our present paper can also be contrasted with recent work of Gelineau and Zeng [11], who present an alternative approach to the combinatorics of the Jacobi–Stirling numbers. In a parallel development, Mongelli has recently established the total positivity of the Jacobi–Stirling numbers in [15]. In addition, in the recent manuscript [16] Mongelli shows that the Jacobi–Stirling numbers are specializations of the elementary and complete homogeneous symmetric functions; he also obtains combinatorial interpretations of a wide class of numbers which include the Jacobi–Stirling numbers as special cases.

The contents of this paper are as follows. In Section 2, we briefly review the Jacobi–Stirling numbers from the original context of left-definite theory. Section 3 deals with a comparison of various properties of the classical Stirling numbers of the second kind and the Jacobi–Stirling numbers. The proofs of many of these properties are similar to the proofs given in [3] so the proofs in this section will either be brief or omitted completely. In Section 4 we give a combinatorial interpretation of the Jacobi–Stirling numbers in terms of set partitions with a prescribed number of blocks. As with the classical Stirling numbers there occur, in a natural way, corresponding (unsigned) Jacobi–Stirling numbers of the first kind (see [8,11,16]). As a result, we sometimes refer to the Jacobi–Stirling numbers as the Jacobi–Stirling numbers of the second kind. In Section 5 we study the Jacobi–Stirling numbers of the first kind and prove several properties of these numbers which are analogues of properties of the classical Stirling numbers of the first kind. In particular, we prove a reciprocity result connecting the Jacobi–Stirling numbers and the Jacobi–Stirling numbers of the first kind. In Section 6 we give two combinatorial interpretations of the Jacobi–Stirling numbers of the first kind in terms of ordered pairs of permutations with prescribed numbers of cycles.

Table 1
The Jacobi–Stirling numbers.

n/j	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$n=0$	1	0	0	0	0	0
$n=1$	0	1	0	0	0	0
$n=2$	0	2γ	1	0	0	0
$n=3$	0	$4\gamma^2$	$6\gamma+2$	1	0	0
$n=4$	0	$8\gamma^3$	$28\gamma^2+20\gamma+4$	$12\gamma+8$	1	0
$n=5$	0	$16\gamma^4$	$120\gamma^3+136\gamma^2+56\gamma+8$	$100\gamma^2+140\gamma+52$	$20\gamma+20$	1

Notation. It is clear from (1.1) that $\left\{{n \atop j}\right\}_{\alpha,\beta}$ is a function of $\alpha+\beta+1$, rather than of α and β individually. With this in mind, we might wish to follow Gelineau and Zeng in setting $z=\alpha+\beta+1$. It turns out, however, that it is more natural for our investigations to view these numbers as a function of γ where $\gamma=(z+1)/2$. Hence, we write $\left\{{n \atop j}\right\}_{\gamma}$ to denote the Jacobi–Stirling number $\left\{{n \atop j}\right\}_{\alpha,\beta}$, where $2\gamma-1=\alpha+\beta+1$. With this notational change, we see from (1.1) that

$$\left\{{n \atop j}\right\}_{\gamma} := \sum_{r=0}^{j}(-1)^{r+j}\frac{(2r+2\gamma-1)\Gamma(r+2\gamma-1)[r(r+2\gamma-1)]^n}{r!(j-r)!\Gamma(j+r+2\gamma)}. \tag{1.5}$$

Observe that the numbers $\left\{{n \atop j}\right\}_{1}$ (that is, $\gamma=1$) are precisely the Legendre–Stirling numbers. Table 1 lists some Jacobi–Stirling numbers for small values of n and j.

A similar table, with the Jacobi–Stirling numbers in terms of α and β, can be found in [10]. In [11], the authors have a table of Jacobi–Stirling numbers given in terms of z; compare also to the results in the contribution [4] on Dowling lattices and r-Whitney numbers. We show below, in Theorem 3.1(iii), that the Jacobi–Stirling numbers satisfy a certain triangular recurrence relation that allows for a fast computation of these numbers.

As with the classical Stirling numbers there occur, in a natural way, corresponding (unsigned) Jacobi–Stirling numbers $\left[{n \atop j}\right]_{\gamma}$ of the first kind (see Section 5 below; see also [8,11,16]). In view of this, we occasionally will call $\left\{{n \atop j}\right\}_{\gamma}$ the *Jacobi–Stirling numbers of the second kind*.

For the remainder of this manuscript, we use the notation $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

2. Background

We give a brief account in this section on the origin of the Jacobi–Stirling numbers. They were discovered in a certain spectral study of the Jacobi differential expression (1.3); for specific details see [10].

When

$$\lambda=\lambda_r=r(r+\alpha+\beta+1)+k \quad (r\in\mathbb{N}_0),$$

one classic solution of the Jacobi differential equation (1.2) is

$$y(x)=P_r^{(\alpha,\beta)}(x)=\binom{r+\alpha}{r}F\left(-r,1+\alpha+\beta+r;1+\alpha;\frac{1}{2}(1-x)\right) \quad (r\in\mathbb{N}_0), \tag{2.1}$$

where $P_r^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree r as defined in the classic text of Szegő [17, Chapter IV].

The most natural setting for an analytic study of (1.2) is the Hilbert function space

$$L^2\big((-1,1);w_{\alpha,\beta}(x)\big) := L^2_{\alpha,\beta}(-1,1),$$

defined by

$$L^2_{\alpha,\beta}(-1,1) := \left\{ f\colon (-1,1) \to \mathbb{C} \;\middle|\; f \text{ is Lebesgue measurable and } \int_{-1}^{1} |f|^2 w_{\alpha,\beta}\,dx < \infty \right\}, \tag{2.2}$$

with inner product

$$(f,g)_{\alpha,\beta} := \int_{-1}^{1} f(x)\bar{g}(x) w_{\alpha,\beta}(x)\,dx \quad \bigl(f, g \in L^2_{\alpha,\beta}(-1,1)\bigr). \tag{2.3}$$

In fact, the Jacobi polynomials $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ form a complete orthogonal sequence in $L^2_{\alpha,\beta}(-1,1)$. Furthermore, in this space (called the right-definite spectral setting), there is a self-adjoint operator $A^{(\alpha,\beta)}$ in $L^2_{\alpha,\beta}(-1,1)$, generated by $\ell_{\alpha,\beta}[\cdot]$, that has the Jacobi polynomials $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ as eigenfunctions. Special cases of these polynomials include the Legendre polynomials ($\alpha = \beta = 0$), the Chebyshev polynomials of the first kind ($\alpha = \beta = -1/2$), the Chebyshev polynomials of the second kind ($\alpha = \beta = 1/2$), and the ultraspherical or Gegenbauer polynomials ($\alpha = \beta$).

The operator $A^{(\alpha,\beta)}$ is an unbounded operator but it is bounded below by kI, where I denotes the identity operator, in $L^2_{\alpha,\beta}(-1,1)$. Consequently, for $k > 0$, a general left-definite operator theory developed by Littlejohn and Wellman [14] applies. In particular, there is a continuum of Hilbert spaces $\{H_t^{(\alpha,\beta)}\}_{t>0}$ where $H_t^{(\alpha,\beta)}$ is called the tth left-definite Hilbert space associated with the pair $(L^2_{\alpha,\beta}(-1,1), A^{(\alpha,\beta)})$. From the viewpoint of the general theory of orthogonal polynomials, it is remarkable that the Jacobi polynomials $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ form a complete orthogonal set in $H_t^{(\alpha,\beta)}$ for each $t > 0$. The upshot of the left-definite analysis of (1.3) is that, for $n \in \mathbb{N}$, the integral composite power $\ell^n_{\alpha,\beta}[\cdot]$ generates the nth left-definite inner product $(\cdot,\cdot)_n^{(\alpha,\beta)}$. In [10], the authors prove the following result which is the key prerequisite to establishing the left-definite theory of the Jacobi differential expression; it is in this result where the Jacobi–Stirling numbers are first introduced.

Theorem 2.1. *Let $n \in \mathbb{N}$. The nth composite power of the Jacobi differential expression (1.3), in Lagrangian symmetric form, is given by*

$$\ell^n_{\alpha,\beta}[y](x) = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=0}^{n} (-1)^j \bigl(c_j^{(\gamma)}(n,k)(1-x)^{\alpha+j}(1+x)^{\beta+j} y^{(j)}(x)\bigr)^{(j)}$$
$$\bigl(x \in (-1,1)\bigr), \tag{2.4}$$

where the coefficients $c_j^{(\gamma)}(n,k)$ $(j = 0,1,\ldots n)$ are nonnegative and given by

$$c_0^{(\gamma)}(n,k) := \begin{cases} 0 & \text{if } k = 0, \\ k^n & \text{if } k > 0, \end{cases}$$

and

$$c_j^{(\gamma)}(n,k) := \begin{cases} \left\{ {n \atop j} \right\}_\gamma & \text{if } k = 0, \\ \sum_{r=0}^{n-j} \binom{n}{r} \left\{ {n-r \atop j} \right\}_\gamma k^r & \text{if } k > 0. \end{cases}$$

In particular, when $k = 0$,

$$\ell^n_{\alpha,\beta}[y](x) = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=1}^{n} (-1)^j \left(\left\{ {n \atop j} \right\}_\gamma (1-x)^{\alpha+j}(1+x)^{\beta+j} y^{(j)}(x) \right)^{(j)}$$
$$\bigl(x \in (-1,1)\bigr). \tag{2.5}$$

Furthermore, $\left\{ {n \atop j} \right\}_\gamma$ *is the coefficient of* x^{n-j} *in the Maclaurin series expansion of*

$$\prod_{m=1}^{j} \frac{1}{1-m(m+2\gamma-1)x} \quad \left(|x| < \frac{1}{j(j+2\gamma-1)}\right). \tag{2.6}$$

From (2.6), we see that we can extend the definition of these numbers to include the initial conditions

$$\left\{ {n \atop 0} \right\}_\gamma = \delta_{n,0} \quad \text{and} \quad \left\{ {0 \atop j} \right\}_\gamma = \delta_{j,0}. \tag{2.7}$$

The original motivation in the discovery of the Jacobi–Stirling numbers can be seen from the rows in Table 1. Indeed, the numbers in the nth row of Table 1 are precisely the coefficients of the nth power of the Jacobi differential expression $\ell_{\alpha,\beta}[\cdot]$; for example,

$$\begin{aligned} w_{\alpha,\beta}(x)\ell^3_{\alpha,\beta}[y](x) = &-\mathbf{1}\big((1-x)^{\alpha+3}(1+x)^{\beta+3}y'''(x)\big)''' \\ &+\big((\mathbf{6}\gamma+\mathbf{2})(1-x)^{\alpha+2}(1+x)^{\beta+2}y''(x)\big)'' \\ &-\big(\mathbf{4}\gamma^2(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\big)'. \end{aligned}$$

On the other hand, the columns in Table 1 demonstrate the rational generating function (2.6) for the Jacobi–Stirling numbers. For example, reading along the column beginning with $j=2$, we see that

$$\begin{aligned} \prod_{r=1}^{2} \frac{1}{1-r(r+2\gamma-1)t} = &\,\mathbf{1} + (\mathbf{6}\gamma+\mathbf{2})t + \big(\mathbf{28}\gamma^{\mathbf{2}}+\mathbf{20}\gamma+\mathbf{4}\big)t^2 \\ &+ \big(\mathbf{120}\gamma^{\mathbf{3}}+\mathbf{136}\gamma^{\mathbf{2}}+\mathbf{56}\gamma+\mathbf{8}\big)t^3 + \cdots. \end{aligned}$$

3. Jacobi Stirling numbers versus Stirling numbers of the second kind

The Jacobi–Stirling numbers $\left\{ {n \atop j} \right\}_\gamma$ are similar in many ways to the classical Stirling numbers of the second kind, which we denote by $\left\{ {n \atop j} \right\}$. As a first point of comparison, we note that, as reported in [14] (see also [9]), the Stirling numbers of the second kind are the coefficients of the integral powers of the second-order Laguerre differential expression $m[\cdot]$, defined by

$$m[y](x) := \frac{1}{x^\alpha e^{-x}}\big(-x^{\alpha+1}e^{-x}y'(x)\big)' \quad \big(x \in (0,\infty)\big).$$

Indeed, for each $n \in \mathbb{N}$,

$$m^n[y](x) = \frac{1}{x^\alpha e^{-x}} \sum_{j=1}^{n} (-1)^j \left(\left\{ {n \atop j} \right\} x^{\alpha+j} e^{-x} y^{(j)}(x) \right)^{(j)}; \tag{3.1}$$

compare (2.5) and (3.1).

Table 2 lists various properties of the Stirling numbers of the second kind.

Regarding the ‘forward difference’ entry in Table 2, recall that the forward difference of a sequence of numbers $\{x_n\}_{n=0}^\infty$ is the sequence $\{\Delta x_n\}_{n=0}^\infty$ defined by

$$\Delta x_n := x_{n+1} - x_n \quad (n \in \mathbb{N}_0).$$

Higher-order forward differences are defined recursively by

$$\Delta^k x_n := \Delta\big(\Delta^{k-1}x_n\big) = \sum_{m=0}^{k} \binom{k}{m} (-1)^m x_{n+k-m}.$$

Table 2
Properties of the classical Stirling numbers of the second kind.

Property	Stirling numbers 2nd kind
Rational GF	$\prod_{r=1}^{j} \frac{1}{1-rx} = \sum_{n=0}^{\infty} \left\{ {n \atop j} \right\} x^{n-j}$
Vertical RR	$\left\{ {n \atop j} \right\} = \sum_{r=j}^{n} \left\{ {r-1 \atop j-1} \right\} j^{n-r}$
Triangular RR	$\left\{ {n \atop j} \right\} = \left\{ {n-1 \atop j-1} \right\} + j \left\{ {n-1 \atop j} \right\}$
	$\left\{ {n \atop 0} \right\} = \delta_{n,0}$; $\left\{ {0 \atop j} \right\} = \delta_{j,0}$
Horizontal GF	$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} (x)_j$, where $(x)_j = x(x-1)\ldots(x-j+1)$
Forward Differences	$\Delta^k(\left\{ {n \atop j} \right\}) \geqslant 0$ for $(n \geqslant j$ and $k \in \mathbb{N}_0)$

Comtet [7, Proposition, p. 749] develops the forward differences inequalities for the Stirling numbers.

We now state a theorem that has comparable properties of the Jacobi–Stirling numbers; the reader will immediately observe the close similarities between Stirling numbers of the second kind and Jacobi–Stirling numbers. The details and proofs of most of these properties are given in [3] for the case of the Legendre–Stirling numbers ($\alpha = \beta = 0$). Since the proofs are almost identical, we will only sketch proofs when necessary.

Theorem 3.1. *The Jacobi–Stirling numbers* $\left\{ {n \atop j} \right\}_\gamma$ *have the following properties.*

(i) (*Rational generating function.*) *For all* $j \in \mathbb{N}$,

$$\prod_{r=1}^{j} \frac{1}{1 - r(r+2\gamma-1)x} = \sum_{n=0}^{\infty} \left\{ {n \atop j} \right\}_\gamma x^{n-j} = \sum_{n=j}^{\infty} \left\{ {n \atop j} \right\}_\gamma x^{n-j} \quad \left(|x| < \frac{1}{j(j+2\gamma-1)}\right);$$

in particular, for each $n, j \in \mathbb{N}$, $\left\{ {n \atop j} \right\}_\gamma$ *is a polynomial in* γ *with nonnegative integer coefficients.*

(ii) (*Vertical recurrence relation.*) *For all* $n, j \in \mathbb{N}_0$,

$$\left\{ {n \atop j} \right\}_\gamma = \sum_{r=j}^{n} \left\{ {r-1 \atop j-1} \right\}_\gamma \big(j(j+2\gamma-1)\big)^{n-r}.$$

(iii) (*Triangular recurrence relation.*) *For all* $n, j \in \mathbb{N}$,

$$\left\{ {n \atop j} \right\}_\gamma = \left\{ {n-1 \atop j-1} \right\}_\gamma + j(j+2\gamma-1) \left\{ {n-1 \atop j} \right\}_\gamma,$$

and for all $n, j \in \mathbb{N}_0$,

$$\left\{ {n \atop 0} \right\}_\gamma = \delta_{n,0}, \qquad \left\{ {0 \atop j} \right\}_\gamma = \delta_{j,0}.$$

(iv) (*Horizontal generating function.*) *For all* $n \in \mathbb{N}_0$,

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_\gamma \langle x \rangle_j^{(\gamma)},$$

where $\langle x \rangle_j^{(\gamma)}$ *is the generalized falling factorial defined by*

$$\langle x \rangle_j^{(\gamma)} := \begin{cases} 1 & \text{if } j = 0, \\ \prod_{m=0}^{j-1} (x - m(m+2\gamma-1)) & \text{if } j \in \mathbb{N}. \end{cases} \tag{3.2}$$

(v) (*Forward differences.*) *For all* $k \in \mathbb{N}_0$ (*and* Δ *acting on the variable* n),

$$\Delta^k\left(\frac{\left\{{n \atop j}\right\}_\gamma}{(2\gamma)^n}\right) \geqslant 0 \quad (n \geqslant j).$$

Proof. The rational generating function, given in (i), is discussed in Theorem 2.1; the complete proof of (i) is given in [10, Theorem 4.1]; we also refer to [11, Section 4.2] where the authors determine this rational generating function in a different and simple way. The vertical recurrence relation in (ii) follows from (i); in fact, the proof is similar to that given in the Legendre–Stirling case which can be found in [3, Theorem 5.2]. Since the triangular recurrence relation is important in the combinatorial interpretation of the Jacobi–Stirling numbers, which we discuss in the next section, we give a proof of (iii). The initial conditions given in (iii) are part of the definition of $\left\{{n \atop j}\right\}_\gamma$, given in (2.7). From (i), we see that

$$\sum_{n=j-1}^{\infty}\left\{{n \atop j-1}\right\}_\gamma x^{n-j+1} = \bigl(1 - j(j+2\gamma-1)x\bigr)\sum_{n=j}^{\infty}\left\{{n \atop j}\right\}_\gamma x^{n-j};$$

shifting the index on the sum on the left-hand side, and carrying out the multiplication on the right-hand side yields

$$\begin{aligned}\sum_{n=j}^{\infty}\left\{{n-1 \atop j-1}\right\}_\gamma x^{n-j} &= \sum_{n=j}^{\infty}\left\{{n \atop j}\right\}_\gamma x^{n-j} - j(j+2\gamma-1)\sum_{n=j}^{\infty}\left\{{n \atop j}\right\}_\gamma x^{n-j+1} \\ &= \sum_{n=j}^{\infty}\left\{{n \atop j}\right\}_\gamma x^{n-j} - j(j+2\gamma-1)\sum_{n=j}^{\infty}\left\{{n-1 \atop j}\right\}_\gamma x^{n-j}\end{aligned}$$

and this implies (iii). The proof of (iv) is similar to the proof of the horizontal generating function for the Legendre–Stirling numbers given in [3, Theorem 5.4]. The proof of (v) is very similar to the proof given in [3, Theorem 5.1] in the case of the Legendre–Stirling numbers. □

For a different approach to parts (i), (iii), and (iv), see [11].

From Table 2 and Theorem 3.1, observe that the rational generating functions for the Stirling numbers of the second kind (which we again note are connected to powers of the Laguerre differential expression in (3.1)) and the Jacobi–Stirling numbers (associated with the powers of the Jacobi differential expression in (2.5)) involve, respectively, the coefficients r and $r(r+\alpha+\beta+1)$ in the denominators of these products. Remarkably, and somewhat mysteriously, these coefficients are, respectively, the eigenvalues that produce the Laguerre and Jacobi polynomial solutions of degree r in, respectively, the Laguerre and Jacobi differential equations. Computing the integral composite powers of both the Laguerre and Jacobi differential equations is entirely *algebraic*, and one would not initially expect these calculations to involve spectral theory. Furthermore, each self-adjoint operator in $L^2_{\alpha,\beta}(-1,1)$ generated by the Jacobi differential expression $\ell_{\alpha,\beta}[\cdot]$ has a discrete (eigenvalues) spectrum only. It is natural to ask why the rational generating function specifically involves the eigenvalues $\{r(r+\alpha+\beta+1)\}_{r=0}^{\infty}$ associated with the operator $A^{(\alpha,\beta)}$ (from Section 2) that has eigenfunctions $\{P_r^{(\alpha,\beta)}\}_{r=0}^{\infty}$ and not the set of eigenvalues of another self-adjoint operator. It seems that there is an interesting connection here that deserves further attention.

The last point that we make in this section is a connection between Legendre–Stirling numbers $\left\{{n \atop j}\right\}_1$ and the classical Stirling numbers $\left\{{n \atop j}\right\}$ of the second kind. In [3, Eq. (3.1)], the authors prove that

$$\left\{{n \atop j}\right\}_1 = \frac{1}{(2j)!}\sum_{m=0}^{2j}(-1)^m\binom{2j}{m}\bigl((j-m)(j+1-m)\bigr)^n.$$

It follows that

$$\begin{aligned}
\left\{ {n \atop j} \right\}_1 &= \frac{\partial^{2n}}{(\partial x)^n (\partial y)^n} \frac{1}{(2j)!} \sum_{m=0}^{2j} (-1)^m \binom{2j}{m} e^{(j-m)x+(j+1-m)y} \bigg|_{x=y=0} \\
&= \frac{1}{(2j)!} \frac{\partial^{2n}}{(\partial x)^n (\partial y)^n} e^{jx+(j+1)y} \big(1-e^{-(x+y)}\big)^{2j} \bigg|_{x=y=0} \\
&= \frac{\partial^{2n}}{(\partial x)^n (\partial y)^n} e^{-jx} e^{-(j-1)y} \frac{(e^{x+y}-1)^{2j}}{(2j)!} \bigg|_{x=y=0} \\
&= \frac{\partial^{2n}}{(\partial x)^n (\partial y)^n} e^{-jx} e^{-(j-1)y} \phi_{2j}(x+y) \bigg|_{x=y=0},
\end{aligned}$$

where $\phi_j(\cdot)$ is the vertical generating function for the Stirling numbers of the second kind; that is,

$$\phi_j(t) := \sum_{n=0}^{\infty} \frac{\left\{ {n \atop j} \right\}}{n!} t^n = \frac{(e^t-1)^j}{j!}.$$

Continuing, we find that

$$\begin{aligned}
\left\{ {n \atop j} \right\}_1 &= \frac{\partial^{2n}}{(\partial x)^n (\partial y)^n} e^{-jx} e^{-(j-1)y} \phi_{2j}(x+y) \bigg|_{x=y=0} \\
&= \left(\frac{\partial}{\partial x}\right)^n e^{-jx} \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\left(\frac{\partial}{\partial y}\right)^{n-\nu} e^{-(j-1)y}\right) \left(\left(\frac{\partial}{\partial y}\right)^{\nu} \phi_{2j}(x+y)\right) \bigg|_{x=y=0} \\
&= \sum_{\nu,\mu=0}^{n} \binom{n}{\nu} \binom{n}{\mu} \big(-(j-1)\big)^{n-\nu} (-j)^{n-\mu} \phi_{2j}^{(\nu+\mu)}(0) \\
&= \sum_{\nu,\mu=0}^{n} \binom{n}{\nu} \binom{n}{\mu} \big(-(j-1)\big)^{n-\nu} (-j)^{n-\mu} \left\{ {\nu+\mu \atop 2j} \right\}.
\end{aligned}$$

Using the forward difference operator, this latter formula for the Legendre–Stirling numbers may be written in the compact form

$$\left\{ {n \atop 1} \right\}_1 = \Delta^n \left\{ {n \atop 2} \right\} = 2^{n-1}, \tag{3.3}$$

$$\left\{ {n \atop j} \right\}_1 = (j-1)^n j^n \Delta_x^n \Delta_y^n \frac{\left\{ {x+y \atop 2j} \right\}}{(j-1)^x j^y} \bigg|_{x=y=0} \quad \text{when } j \geqslant 2, \tag{3.4}$$

with the obvious meaning of Δ_x and Δ_y. It would be interesting to see if there is a similar connection between the Stirling numbers of the second kind and the Jacobi–Stirling numbers.

4. A combinatorial interpretation of the Jacobi–Stirling numbers

The Stirling number of the second kind $\left\{ {n \atop j} \right\}$ is the number of set partitions of $\{1, 2, \dots, n\}$ into j nonempty blocks. That is, $\left\{ {n \atop j} \right\}$ is the number of ways of placing n objects into j nonempty, indistinguishable sets; for a full account of their properties, see [1] and Comtet [6, Chapter V]. With this in mind, it is natural to ask for a similar combinatorial interpretation of the Jacobi–Stirling number $\left\{ {n \atop j} \right\}_\gamma$. Indeed, Andrews and Littlejohn [2] have generalized the notion of a set partition to give a combinatorial interpretation of the Legendre–Stirling number $\left\{ {n \atop j} \right\}_1$; one might call Andrews and Littlejohn's generalized set partitions *Legendre–Stirling set partitions*. More recently, Gelineau and Zeng [11] have

Table 3
Jacobi–Stirling set partitions.

Zero blocks	Nonzero blocks	Zero blocks	Nonzero blocks
$\varnothing, \varnothing, \varnothing$	$\{1_1, 1_2, 3_1\}, \{2_1, 2_2, 3_2\}$	$\{3_1\}, \varnothing, \varnothing$	$\{1_1, 1_2, 3_2\}, \{2_1, 2_2\}$
$\varnothing, \varnothing, \varnothing$	$\{1_1, 1_2, 3_1\}, \{2_1, 2_2, 3_2\}$	$\varnothing, \varnothing, \{3_2\}$	$\{1_1, 1_2, 3_1\}, \{2_1, 2_2\}$
$\varnothing, \varnothing, \{3_1\}$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_2\}$	$\varnothing, \{3_2\}, \varnothing$	$\{1_1, 1_2, 3_1\}, \{2_1, 2_2\}$
$\varnothing, \{3_1\}, \varnothing$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_2\}$	$\{3_2\}, \varnothing, \varnothing$	$\{1_1, 1_2, 3_1\}, \{2_1, 2_2\}$
$\{3_1\}, \varnothing, \varnothing$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_2\}$	$\{2_1\}, \varnothing, \varnothing$	$\{1_1, 1_2, 2_2\}, \{3_1, 3_2\}$
$\varnothing, \varnothing, \{3_2\}$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_1\}$	$\varnothing, \{2_1\}, \varnothing$	$\{1_1, 1_2, 2_2\}, \{3_1, 3_2\}$
$\varnothing, \{3_2\}, \varnothing$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_1\}$	$\varnothing, \varnothing, \{2_1\}$	$\{1_1, 1_2, 2_2\}, \{3_1, 3_2\}$
$\{3_2\}, \varnothing, \varnothing$	$\{1_1, 1_2\}, \{2_1, 2_2, 3_1\}$	$\{2_2\}, \varnothing, \varnothing$	$\{1_1, 1_2, 2_1\}, \{3_1, 3_2\}$
$\varnothing, \varnothing, \{3_1\}$	$\{1_1, 1_2, 3_2\}, \{2_1, 2_2\}$	$\varnothing, \{2_2\}, \varnothing$	$\{1_1, 1_2, 2_1\}, \{3_1, 3_2\}$
$\varnothing, \{3_1\}, \varnothing$	$\{1_1, 1_2, 3_2\}, \{2_1, 2_2\}$	$\varnothing, \varnothing, \{2_2\}$	$\{1_1, 1_2, 2_1\}, \{3_1, 3_2\}$

found a statistic on Legendre–Stirling set partitions which allows them to interpret $\left\{ {n \atop j} \right\}_\gamma$ as a generating function over Legendre–Stirling set partitions. In this section we generalize Legendre–Stirling set partitions still further, to obtain objects we will call *Jacobi–Stirling set partitions*. When $\gamma = 1$ it will be clear that the Jacobi–Stirling set partitions are in fact Legendre–Stirling set partitions, and we will show that for any positive integer γ, the Jacobi–Stirling numbers count Jacobi–Stirling set partitions.

To describe a combinatorial interpretation of the Jacobi–Stirling number $\left\{ {n \atop j} \right\}_\gamma$, let $[n]_2$ denote the set $\{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$, which contains two copies of each positive integer between 1 and n; we may say that these are the integers $\{1, 2, \ldots, n\}$ with two different colors. By convention $[0]_2$ is the empty set.

Definition 4.1. For all $n, j, \gamma \in \mathbb{N}_0$, a *Jacobi–Stirling set partition* of $[n]_2$ into γ zero blocks and j nonzero blocks is an ordinary set partition of $[n]_2$ into $j + \gamma$ blocks for which the following conditions hold:

(1) γ of our blocks, called the *zero blocks*, are distinguishable, but all other blocks are indistinguishable.
(2) The zero blocks may be empty, but all other blocks are nonempty.
(3) The union of the zero blocks may not contain both copies of any number.
(4) Each nonzero block contains both copies of the smallest number it contains, but does not contain both copies of any other number.

Example 4.1. As we see in Table 3, there are 20 Jacobi–Stirling set partitions of $[3]_2$ into $\gamma = 3$ zero blocks and $j = 2$ nonzero blocks.

As we show next, Jacobi–Stirling numbers count Jacobi–Stirling set partitions.

Theorem 4.1. *For all $n, j, \gamma \in \mathbb{N}_0$, the Jacobi–Stirling number $\left\{ {n \atop j} \right\}_\gamma$ is the number of Jacobi–Stirling set partitions of $[n]_2$ into γ zero blocks and j nonzero blocks.*

Proof. For all $n, j, \gamma \in \mathbb{N}_0$, let $P(n, j, \gamma)$ denote the number of Jacobi–Stirling set partitions of $[n]_2$ into γ zero blocks and j nonzero blocks. Since the Jacobi–Stirling numbers are determined by the initial conditions and recurrence relation in Theorem 3.1(iii), it is sufficient to show that $P(n, j, \gamma)$ satisfies the same initial conditions and recurrence relation.

Since the union of the zero blocks of a Jacobi–Stirling set partition cannot contain both 1_1 and 1_2, we see that $P(n, 0, \gamma) = 0$ if $n > 0$. On the other hand, for each γ there is one Jacobi–Stirling set partition of the empty set, so $P(0, 0, \gamma) = 1$. Finally, since the nonzero blocks of a Jacobi–Stirling set partition must be nonempty, we see that $P(0, j, \gamma) = 0$ if $j > 0$. Therefore $P(n, j, \gamma)$ satisfies the same initial conditions as $\left\{ {n \atop j} \right\}_\gamma$.

To see that $P(n, j, \gamma)$ satisfies the same recurrence relation, we note that for $n \geqslant 1$, Jacobi–Stirling set partitions of $[n]_2$ into γ zero blocks and j nonzero blocks come in two disjoint types:

(i) those in which n_1 and n_2 are in the same block;
(ii) those in which n_1 and n_2 are in different blocks.

Each Jacobi–Stirling set partition in class (i) can be uniquely constructed from a Jacobi–Stirling set partition of $[n-1]_2$ into γ zero blocks and $j-1$ nonzero blocks by appending a nonzero block containing only n_1 and n_2. Therefore there are $P(n-1, j-1, \gamma)$ partitions in class (i). On the other hand, each Jacobi–Stirling set partition in class (ii) can be uniquely constructed from a Jacobi–Stirling set partition of $[n-1]_2$ into γ zero blocks and $j-1$ nonzero blocks by either inserting n_1 into a zero block and inserting n_2 into a nonzero block, or by inserting n_1 into a nonzero block and inserting n_2 into any block not containing n_1. There are $\gamma j P(n-1, j, \gamma)$ partitions of the first type, and $j(\gamma + j - 1)P(n-1, j, \gamma)$ partitions of the second type, so there are $j(j + 2\gamma - 1)P(n-1, j, \gamma)$ partitions in class (ii). Therefore $P(n, j, \gamma) = P(n-1, j-1, \gamma) + j(j + 2\gamma - 1)P(n-1, j, \gamma)$, so $P(n, j, \gamma)$ satisfies the same recurrence relation as $\left\{ {n \atop j} \right\}_\gamma$, and the result follows. □

Example 4.2. In [2, Example 4.4], the authors showed that $\left\{ {n \atop 1} \right\}_1 = 2^{n-1}$. This argument generalizes to show that $\left\{ {n \atop 1} \right\}_\gamma = (2\gamma)^{n-1}$ for all $n \in \mathbb{N}$.

Example 4.3. In this example we give a direct combinatorial proof that $\left\{ {n \atop n-1} \right\}_\gamma = 2\binom{n}{3} + 2\gamma\binom{n}{2}$, where $\binom{n}{j}$ denotes the usual binomial coefficient. By Theorem 4.1, the quantity $\left\{ {n \atop n-1} \right\}_\gamma$ is the number of Jacobi–Stirling partitions of $[n]_2$ into γ zero blocks and $n-1$ nonzero blocks. In such a partition there must be exactly one number k which is not the minimum in its block, and at least one copy of k is in a nonzero block. If both copies of k are in nonzero blocks, then we may construct our partition uniquely by choosing the elements $i < j < k$ of these blocks, and then placing k_1 and k_2 in blocks with minima i and j. Thus there are $2\binom{n}{3}$ of these partitions. Alternatively, if one copy of k is in a zero block, then we may construct our partition uniquely by choosing the elements $i < k$ of the nonzero block containing k, constructing that block with one copy of k, and placing the other copy of k in one of the γ zero blocks. Therefore there are $2\gamma\binom{n}{2}$ of these partitions. Combining these two counts, we find that $\left\{ {n \atop n-1} \right\}_\gamma = 2\binom{n}{3} + 2\gamma\binom{n}{2}$, as claimed.

In a development independent of this work, Mongelli has recently given another combinatorial interpretation of the Jacobi–Stirling numbers [16]. In fact, Mongelli gives a combinatorial interpretation of a general family of numbers, which includes the Jacobi–Stirling numbers as a special case. Translated into our setting, Mongelli shows inductively that if $z = 2\gamma - 1$ is a positive integer then $\left\{ {n \atop j} \right\}_\gamma$ is the number of set partitions of $[n]_2$ into z zero blocks and j nonzero blocks for which the following conditions hold.

(1) The zero blocks are distinguishable, but the nonzero blocks are indistinguishable.
(2) The zero blocks may be empty, but the nonzero blocks are nonempty.
(3) No zero block may contain the first copy of any number.
(4) Each nonzero block contains both copies of the smallest number it contains.

For convenience, we call one of Mongelli's set partitions a *long Jacobi–Stirling set partition*.

For any $n, j, \gamma \in \mathbb{N}_0$, we can give a bijection between the associated Jacobi–Stirling set partitions and the associated long Jacobi–Stirling set partitions. To do this, suppose a given long Jacobi–Stirling set partition has zero blocks $Z_0, Z_1, \ldots, Z_{2\gamma-2}$ and nonzero blocks $B_1, \ldots, B_j$. To obtain a Jacobi–Stirling set partition, for each i, $1 \leqslant i \leqslant n$, we do the following.

- If i_1 and i_2 are in different nonzero blocks, then we leave i_1 and i_2 where they are.

Table 4
The Jacobi–Stirling numbers of the first kind.

n/j	0	$j=1$	$j=2$	$j=3$	$j=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	2γ	1	0	0
$n=3$	0	$8\gamma^2+4\gamma$	$6\gamma+2$	1	0
$n=4$	0	$48\gamma^3+72\gamma^2+24\gamma$	$44\gamma^2+52\gamma+12$	$12\gamma+8$	1
$n=5$	0	$384\gamma^4+1152\gamma^3+1056\gamma^2+288\gamma$	$400\gamma^3+1016\gamma^2+744\gamma+144$	$140\gamma^2+260\gamma+108$	$20\gamma+20$

- If i_1 and i_2 are in the same nonzero block and they are the smallest element of that block, then we leave i_1 and i_2 where they are.
- If i_1 and i_2 are in the same nonzero block B_k and they are not the smallest element of that block, then we put i_1 in Z_0 and we leave i_2 where it is.
- If $i_2 \in Z_k$ for $0 \leqslant k \leqslant \gamma - 1$ then we leave i_1 and i_2 where they are.
- If $i_2 \in Z_{\gamma+k-1}$ for $1 \leqslant k \leqslant \gamma - 1$ and $i_1 \in B_m$ then we put i_2 in B_m and we put i_1 in Z_k.

Now $Z_\gamma, Z_{\gamma+1}, \dots, Z_{2\gamma-2}$ are empty, so we remove them from our set partition. The result is a Jacobi–Stirling set partition, and we can reverse the process step-by-step to recover the original long Jacobi–Stirling set partition.

5. The Jacobi–Stirling numbers of the first and second kinds

We saw by comparing Theorem 3.1 with Table 2 that the Jacobi–Stirling numbers are natural analogues of the Stirling numbers of the second kind (see also [11] and [16]). In the case of the Stirling numbers we can invert the horizontal generating function given in Table 2 to obtain the Stirling numbers of the first kind. In particular, these (unsigned) Stirling numbers of the first kind $\left[{n \atop j}\right]$ may be defined by the identity

$$(x)_n = \sum_{j=0}^{n} (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix} x^j \quad (n \in \mathbb{N}), \tag{5.1}$$

where $(x)_n$ is the falling factorial given in Table 2.

Similarly, if we invert the horizontal generating function for the Jacobi–Stirling numbers, given in (iv) of Theorem 3.1, we obtain a collection of numbers we will call the (unsigned) *Jacobi–Stirling numbers of the first kind*, and which we will denote by $\left[{n \atop j}\right]_\gamma$. Specifically,

$$\langle x \rangle_n^{(\gamma)} = \sum_{j=0}^{n} (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix}_\gamma x^j, \tag{5.2}$$

where $\langle x \rangle_j^{(\gamma)}$ is the generalized falling factorial defined in (3.2). We immediately obtain the following biorthogonality relationships:

$$\sum_{m \leqslant j \leqslant n} (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix}_\gamma \begin{Bmatrix} j \\ m \end{Bmatrix}_\gamma = \delta_{n,m} \quad \text{and} \quad \sum_{m \leqslant j \leqslant n} (-1)^{j+m} \begin{Bmatrix} n \\ j \end{Bmatrix}_\gamma \begin{bmatrix} j \\ m \end{bmatrix}_\gamma = \delta_{n,m} \quad (n, m \in \mathbb{N}_0).$$

Table 4 lists the Jacobi–Stirling numbers of the first kind for small n and j; see also Table 2 in [11].

Like the Stirling numbers of the second kind, the unsigned Stirling numbers of the first kind satisfy a triangular recurrence relation (see [6, p. 214]): for all $n, j \in \mathbb{N}$ we have

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ j \end{bmatrix}.$$

As we show next, the Jacobi–Stirling numbers of the first kind satisfy a similar triangular recurrence relation.

Theorem 5.1. *For all $n, j \in \mathbb{N}_0$ we have*

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{n,0} \quad \text{and} \quad \begin{bmatrix} 0 \\ j \end{bmatrix}_\gamma = \delta_{j,0}, \tag{5.3}$$

and for all $n, j \in \mathbb{N}$ we have

$$\begin{bmatrix} n \\ j \end{bmatrix}_\gamma = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_\gamma + (n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma. \tag{5.4}$$

Proof. Line (5.3) is immediate from (5.2), since $\langle x\rangle_0^{(\gamma)} = 1$ and for all $j \in \mathbb{N}$ the polynomial $\langle x\rangle_j^{(\gamma)}$ is divisible by x.

To obtain (5.4), first note that the coefficient of x^j on the right side of (5.2) is $(-1)^{j+n}\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_\gamma$. Now observe that if $n \in \mathbb{N}$ then

$$\begin{aligned}\langle x\rangle_n^{(\gamma)} &= \big(x-(n-1)(n+2\gamma-2)\big)\langle x\rangle_{n-1}^{(\gamma)} \\ &= \big(x-(n-1)(n+2\gamma-2)\big)\sum_{j=0}^{n-1}(-1)^{n-1+j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma x^j,\end{aligned}$$

so the coefficient of x^j on the left side of (5.2) is

$$(-1)^{j+n}(n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma + (-1)^{j+n}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_\gamma.$$

We obtain (5.4) when we equate these two expressions for the coefficient of x^j. □

Next in this section, we prove a reciprocity result which connects Jacobi–Stirling numbers of the two kinds. To state this result, first observe that there is a unique collection $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_\gamma$ $(n, j \in \mathbb{Z})$ of polynomials in γ satisfying the initial condition

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{n,0}, \qquad \begin{bmatrix} 0 \\ j \end{bmatrix}_\gamma = \delta_{j,0}, \tag{5.5}$$

and recurrence relation

$$\begin{bmatrix} n \\ j \end{bmatrix}_\gamma = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_\gamma + (n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma \quad (n, k \in \mathbb{Z}). \tag{5.6}$$

Moreover, these polynomials are the Jacobi–Stirling numbers of the first kind when $n, j \in \mathbb{N}$. Similarly, there is a unique collection $\left\{\begin{smallmatrix} n \\ j \end{smallmatrix}\right\}_\gamma$ $(n, j \in \mathbb{Z})$ of polynomials in γ satisfying the initial condition

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix}_\gamma = \delta_{n,0}, \qquad \begin{Bmatrix} 0 \\ j \end{Bmatrix}_\gamma = \delta_{j,0}, \tag{5.7}$$

and recurrence relation

$$\begin{Bmatrix} n \\ j \end{Bmatrix}_\gamma = \begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix}_\gamma + j(j+2\gamma-1)\begin{Bmatrix} n-1 \\ j \end{Bmatrix}_\gamma \quad (n, k \in \mathbb{Z}). \tag{5.8}$$

Moreover, these polynomials are the Jacobi–Stirling numbers of the second kind when $n, j \in \mathbb{N}$. It is not difficult to show that if $n \neq 0$ and $j \neq 0$ differ in sign then $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_\gamma = \left\{\begin{smallmatrix} n \\ j \end{smallmatrix}\right\}_\gamma = 0$, but we might hope that for various negative n and j we obtain interesting new polynomials $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_\gamma$ and $\left\{\begin{smallmatrix} n \\ j \end{smallmatrix}\right\}_\gamma$. Our reciprocity result, which is an analogue of a similar result [13, line (2.4)] for classical Stirling numbers, shows that we actually recover the Jacobi–Stirling numbers of the second and first kinds.

Theorem 5.2. *For all $n, j \in \mathbb{Z}$ we have*

$$\begin{Bmatrix} -j \\ -n \end{Bmatrix}_{\gamma} = (-1)^{n+j} \begin{bmatrix} n \\ j \end{bmatrix}_{1-\gamma}. \tag{5.9}$$

Proof. The Jacobi–Stirling numbers of the second kind are uniquely determined by (5.7) and (5.8), so it is sufficient to show that the quantities $L(n, j) = (-1)^{n+j}\left[\begin{smallmatrix} -j \\ -n \end{smallmatrix}\right]_{1-\gamma}$ also satisfy (5.7) and (5.8).

The fact that $L(n, j)$ satisfies (5.7) is immediate from (5.5), so suppose $n \neq 0$ and $j \neq 0$. Then we have

$$\begin{aligned} L(n-1, j-1) &= (-1)^{n+j} \begin{bmatrix} -j+1 \\ -n+1 \end{bmatrix}_{1-\gamma} \\ &= (-1)^{n+j} \begin{bmatrix} -j \\ -n \end{bmatrix}_{1-\gamma} + (-1)^{n+j}(-j)\big(-j+1+2(1-\gamma)-2\big) \begin{bmatrix} -j \\ -n+1 \end{bmatrix}_{1-\gamma} \\ &= L(n, j) - j(j+2\gamma-1)L(n-1, j), \end{aligned}$$

and the result follows. $\square$

Concluding this section, we now turn our attention to the unimodality of the Jacobi–Stirling numbers of the first and second kinds. Recall that a sequence of real numbers $\{x_n\}_{n=0}^{\infty}$ is *unimodal* whenever there exists j such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_j$ and $x_j \geqslant x_{j+1} \geqslant \cdots$.

As in [3, Section 5.7], we consider the horizontal generating function

$$A_n(x) = \sum_{j=0}^{n} \begin{Bmatrix} n \\ j \end{Bmatrix}_{\gamma} x^j \quad (n \in \mathbb{N}_0) \tag{5.10}$$

which, by Theorem 3.1(iii), is a polynomial of degree n with leading coefficient $\left\{\begin{smallmatrix} n \\ n \end{smallmatrix}\right\}_{\gamma} = 1$. Again, by this property, we find that $A_0(x) = 1$ and, for $n \geqslant 1$,

$$\begin{aligned} A_n(x) &= \sum_{j=1}^{n} \begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix}_{\gamma} x^j + \sum_{j=1}^{n} j(j+2\gamma-1) \begin{Bmatrix} n-1 \\ j \end{Bmatrix}_{\gamma} x^j \\ &= x\big(A_{n-1}(x) + 2\gamma A'_{n-1}(x) + xA''_{n-1}(x)\big). \end{aligned}$$

For example, we see that

$$A_1(x) = x, \qquad A_2(x) = x(x+2\gamma), \qquad A_3(x) = x\big(x^2 + 2(3\gamma+1)x + 4\gamma^2\big).$$

Since $\gamma > 0$, A_1, A_2, and A_3 are real polynomials, the zeros of which are all real, simple, and non-positive. Along the same lines as the proof of Theorem 5.7 in [3], we obtain

Theorem 5.3. *Let $n \in \mathbb{N}$. The zeros of A_n, defined in (5.10), are real, simple and non-positive. Moreover, $A_n(0) = 0$.*

For a different proof, see [15, Theorem 4]. From Theorem 5.3, Eq. (5.2), and a standard criterion for unimodality as given in Comtet [6, p. 270], we can state the following result.

Theorem 5.4. *The unsigned Jacobi–Stirling numbers of the first kind and the Jacobi–Stirling numbers of the second kind are unimodal with either a peak or a plateau of 2 points.*

6. Two combinatorial interpretations of the Jacobi–Stirling numbers of the first kind

The unsigned classical Stirling number of the first kind $\left[{n \atop j}\right]$ is the number of permutations of $\{1, 2, \dots, n\}$ with j cycles. With this in mind, it is natural to ask for a similar combinatorial interpretation of the Jacobi–Stirling number of the first kind $\left[{n \atop j}\right]_\gamma$. Indeed, Egge [8] has given a combinatorial interpretation of the Legendre–Stirling number $\left[{n \atop j}\right]_1$ in terms of pairs of permutations; one might call these *Legendre–Stirling permutation pairs*. More recently, Gelineau and Zeng [11] have found a statistic on Legendre–Stirling permutation pairs which allows them to interpret $\left[{n \atop j}\right]_\gamma$ as a generating function over these pairs. In this section we generalize Legendre–Stirling permutation pairs still further, to obtain objects we will call *Jacobi–Stirling permutation pairs*. These objects will come in two flavors, balanced and unbalanced. When $\gamma = 1$ it will be clear that both the balanced and the unbalanced Jacobi–Stirling permutation pairs are in fact Legendre–Stirling permutation pairs. In addition, we will show that for any positive integer γ, the Jacobi–Stirling numbers count both the balanced and the unbalanced Jacobi–Stirling permutation pairs. In connection with these pairs, it will be useful to note that the *cycle maxima* of a given permutation are the numbers which are largest in their cycles. For example, if $\pi = (4, 6, 1)(9, 2, 3)(7, 8)$ is a permutation in S_{10}, written in cycle notation, then its cycle maxima are $5, 6, 8, 9$, and 10.

Definition 6.1. Suppose $n, \gamma \in \mathbb{N}_0$. A *balanced Jacobi–Stirling permutation pair* of length n is an ordered pair (π_1, π_2) with $\pi_1 \in S_{n+\gamma}$ and $\pi_2 \in S_{n+\gamma-1}$ for which the following conditions hold:

(1) π_1 has one more cycle than π_2.
(2) The cycle maxima of π_1 which are less than $n + \gamma$ are exactly the cycle maxima of π_2.
(3) For each k which is not a cycle maximum, at least one of $\pi_1(k)$ and $\pi_2(k)$ is less than or equal to n.

In [8] Egge defines a Legendre–Stirling permutation pair of length n to be an ordered pair (π_1, π_2) with $\pi_1 \in S_{n+1}$ and $\pi_2 \in S_n$, such that π_1 has one more cycle than π_2 and the cycle maxima of π_1 which are less than $n + 1$ are exactly the cycle maxima of π_2. If (π_1, π_2) is an ordered pair of permutations with $\pi_1 \in S_{n+1}$ and $\pi_2 \in S_n$ which satisfies conditions (1) and (2) of Definition 6.1, then the fact that $\pi_2 \in S_n$ implies (π_1, π_2) also satisfies condition (3). Therefore, the Legendre–Stirling permutation pairs of length n are exactly the balanced Jacobi–Stirling permutation pairs of length n with $\gamma = 1$. The Legendre–Stirling permutation pairs are counted by the Legendre–Stirling numbers of the first kind $\left[{n \atop j}\right]_1$; as we show next, the balanced Jacobi–Stirling permutation pairs are counted by the Jacobi–Stirling numbers of the first kind.

Theorem 6.1. *For all $n, j, \gamma \in \mathbb{N}_0$, the number of balanced Jacobi–Stirling permutation pairs (π_1, π_2) of length n in which π_1 has exactly $\gamma + j$ cycles is $\left[{n \atop j}\right]_\gamma$.*

Proof. Let $a_{n,j}$ denote the number of balanced Jacobi–Stirling permutation pairs (π_1, π_2) of length n in which π_1 has exactly $\gamma + j$ cycles.

It follows from (5.5) and (5.6) that $\left[{1 \atop j}\right]_\gamma = \delta_{j,1}$. On the other hand, if (π_1, π_2) is a balanced Jacobi–Stirling permutation pair of length 1 in which π_1 has exactly $\gamma + j$ cycles, then we must have $j \leqslant 1$, since $\pi_1 \in S_{1+\gamma}$. If $j = 0$ then some entry of π_1 violates Definition 6.1(3), so we must have $j = 1$. Moreover, when $j = 1$ both π_1 and π_2 must be the identity permutation, so $a_{1,j} = \delta_{j,1}$. Therefore the result holds for $n = 1$.

Now suppose $n > 1$ and the result holds for $n - 1$; we argue by induction on n. To obtain $a_{n,j}$, first observe that by condition (2) of Definition 6.1, if (π_1, π_2) is a balanced Jacobi–Stirling permutation pair of length n then 1 is a fixed point in π_1 if and only if it is a fixed point in π_2. Pairs (π_1, π_2) in which 1 is a fixed point are in bijection with pairs (σ_1, σ_2) of length $n - 1$ in which σ_1 has $j - 1 + \gamma$ cycles by removing the 1 from each permutation and decreasing all other entries by 1. Each pair (π_1, π_2) in which 1 is not a fixed point may be constructed uniquely by choosing a pair

(σ_1, σ_2) of length $n-1$ in which σ_1 has $j+\gamma$ cycles, increasing each entry of each permutation by 1, and inserting 1 after an entry of each permutation. There are $a_{n-1,j}$ pairs (σ_1, σ_2), there are $(n-1)(n+\gamma-2)$ ways to insert 1 so that $\pi_1(1) \leqslant n$, and there are $\gamma(n-1)$ ways to insert 1 so that $\pi_1(1) > n$. Combining these observations and using induction to eliminate $a_{n-1,j-1}$ and $a_{n-1,k}$ we find

$$a_{n,j} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_\gamma + (n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma = \begin{bmatrix} n \\ j \end{bmatrix}_\gamma,$$

as desired. □

We have shown that the Jacobi–Stirling numbers of the first kind count balanced Jacobi–Stirling permutation pairs whenever γ is an integer, but the orthogonal polynomials that give rise to these numbers include interesting special cases in which γ is a half integer. Most notably, the Chebyshev polynomials of the first kind occur when $\gamma = \frac{1}{2}$, while the Chebyshev polynomials of the second kind occur when $\gamma = \frac{3}{2}$. To address the combinatorics of the case in which 2γ is an integer, we introduce unbalanced Jacobi–Stirling permutation pairs, and we show that these pairs are also counted by the Jacobi–Stirling numbers of the first kind.

Definition 6.2. Suppose $n, 2\gamma \in \mathbb{N}_0$. An *unbalanced Jacobi–Stirling permutation pair* of length n is an ordered pair (π_1, π_2) with $\pi_1 \in S_{n+2\gamma-1}$ and $\pi_2 \in S_n$ for which the following hold.

(1) π_1 has $2\gamma - 1$ more cycles than π_2.
(2) The cycle maxima of π_1 which are less than $n+1$ are exactly the cycle maxima of π_2.

It is not difficult to see that when $\gamma = 1$ the unbalanced Jacobi–Stirling permutation pairs are exactly the Legendre–Stirling permutation pairs introduced by Egge, which are counted by the Legendre–Stirling numbers of the first kind $\left[{n \atop j}\right]_1$. As we show next, the unbalanced Jacobi–Stirling permutation pairs are counted by the Jacobi–Stirling numbers of the first kind. We note that this result is a special case of a result proved independently by Mongelli [16].

Theorem 6.2. *For all $n, j, \gamma \in \mathbb{N}_0$, the number of unbalanced Jacobi–Stirling permutation pairs (π_1, π_2) of length n in which π_2 has exactly j cycles is $\left[{n \atop j}\right]_\gamma$.*

Proof. Let $a_{n,j}$ denote the number of unbalanced Jacobi–Stirling permutation pairs (π_1, π_2) of length n in which π_2 has exactly j cycles.

It follows from (5.5) and (5.6) that $\left[{1 \atop j}\right]_\gamma = \delta_{j,1}$. On the other hand, if (π_1, π_2) is an unbalanced Jacobi–Stirling permutation pair of length 1 in which π_2 has exactly j cycles, then we must have $j = 1$, since $\pi_2 \in S_1$. Moreover, when $j = 1$ both π_1 and π_2 must be the identity permutation, so $a_{1,j} = \delta_{j,1}$. Therefore the result holds for $n = 1$.

Now suppose $n > 1$ and the result holds for $n-1$; we argue by induction on n. To obtain $a_{n,j}$, first observe that by condition 2 of Definition 6.2, if (π_1, π_2) is an unbalanced Jacobi–Stirling permutation pair of length n then 1 is a fixed point in π_1 if and only if it is a fixed point in π_2. Pairs (π_1, π_2) in which 1 is a fixed point are in bijection with pairs (σ_1, σ_2) of length $n-1$ in which σ_2 has $j-1$ cycles by removing the 1 from each permutation and decreasing all other entries by 1. Each pair (π_1, π_2) in which 1 is not a fixed point may be constructed uniquely by choosing a pair (σ_1, σ_2) of length $n-1$ in which σ_2 has j cycles, increasing each entry of each permutation by 1, and inserting 1 after an entry of each permutation. There are $a_{n-1,j}$ pairs (σ_1, σ_2), and there are $(n-1)(n+2\gamma-2)$ ways to insert our 1's. Combining these observations and using induction to eliminate $a_{n-1,j-1}$ and $a_{n-1,k}$ we find

$$a_{n,j} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_\gamma + (n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ j \end{bmatrix}_\gamma = \begin{bmatrix} n \\ j \end{bmatrix}_\gamma,$$

as desired. □

We conclude with an example involving unbalanced Jacobi–Stirling permutation pairs.

Example 6.1. In this example we give a direct combinatorial proof that

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_\gamma = (n-1)! \prod_{j=0}^{n-1} (2\gamma + j).$$

By Theorem 6.2, the quantity $\begin{bmatrix} n \\ 1 \end{bmatrix}_\gamma$ is the number of unbalanced Jacobi–Stirling permutation pairs (π_1, π_2) of length n, where $\pi_1 \in S_{n+2\gamma-1}$ has 2γ cycles and $\pi_2 \in S_n$ has 1 cycle. Since π_2 has 1 cycle, there are $(n-1)!$ choices for this permutation. Moreover, the cycle maxima for π_1 must be $n, n+1, \ldots, n+2\gamma-1$. Now we may place 1 in π_2 in 2γ ways, we may place 2 in $2\gamma+1$ ways, and in general we may place $j+1$ in $2\gamma+j$ ways. Thus there are $(n-1)! \prod_{j=0}^{n-1}(2\gamma+j)$ of these unbalanced Jacobi–Stirling permutation pairs.

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